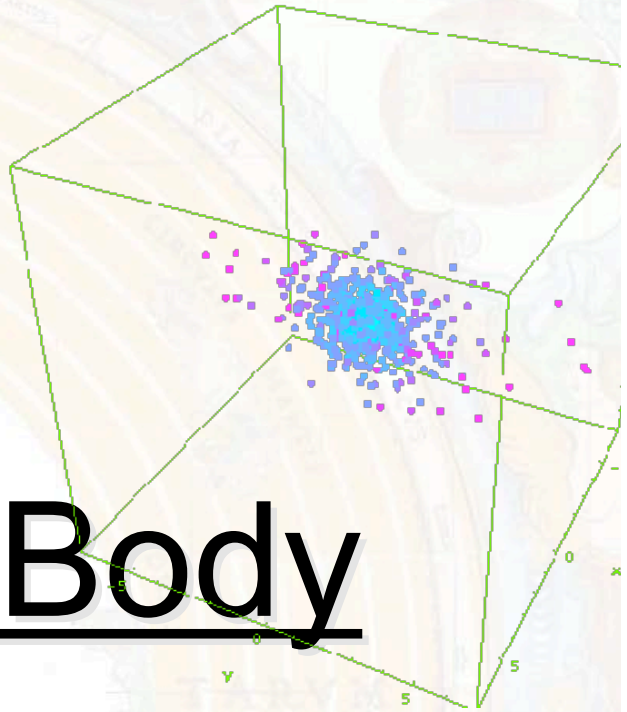
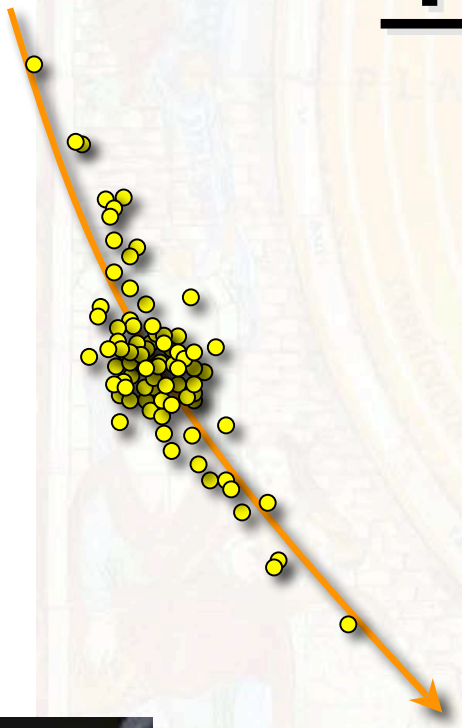


The Art of N-Body Simulations

Luis A. Aguilar
aguilar@astrosen.unam.mx



This school is not about N-body simulations. But rather, about the art of N-body simulations.

To run an N-body simulation you require a working code and a computer.

To do an N-body simulation and actually learn something from it, you also need understanding of the physics and the numerics involved.

After all, computers only give you numbers, but not insight.

Running simulations can actually lead you on the wrong path.
It is only by doing simulations that you can gain understanding.

It is the goal of this school to show you how to do N-body simulations. This is an art that, unfortunately, is usually not taught in books or courses, but it is passed from tutors to students.

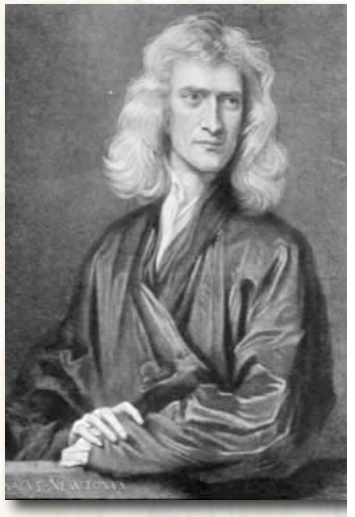
Background

To talk about N-body simulations, we should go back to gravity.

It all began with an apple (or so thay say) and Newton in the XVII century.

"Planetas omnes in se muto graves esse jam ante probavimus,
ut & gravitatem in unumquemque seorsim spectatum
esse reciproce ut quadratum distantiœ locorum a centro planetæ.
Et inde consequens est gravitatem in omnes proportionalem
esse materiæ in iisdem".

Isaac Newton,
Propositio vii, Teorema VII.
Philosophiæ Naturalis Principia Mathematica.

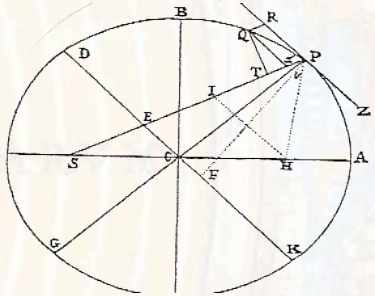


Background

To talk about N-body simulations, we should go back to gravity.

The solution of the two-body problem, lead Newton to the invention of Calculus.
The three-body problem, however, proved intractable. Newton wrote that its solution,

"Unless I am much mistaken, it would exceed the force of human wit to consider so many causes of motion at the same time".



And right there, from the very beginning, laid the seed that would lead to N-body simulations.

Background

The historical development of the field of classical gravitation

The gravitational *N*-body problem (in the classical regime), has a historical development that is inverted with respect to other branches of Physics: the rest has started from a global description of phenomena to develop, subsequently, a local description of the fundamental interaction among the components of the system.

Thermodynamics ➜ *Statistical Mechanics,*
Fluid Mechanics ➜ *Kinetic Theory of gases,*
Bulk properties of materials ➜ *Atomic and molecular structure.*

Background

The historical development of the field of classical gravitation

In the gravitational problem we knew the interaction rule from the beginning.

$$F = -G\frac{m_i m_j}{r_{ij}^2}$$

The difficulty was in applying it to several particles at once.

The study of the *N*-body problem followed two paths: the first relied on finding approximations to the true solution. This gave rise to *Celestial Mechanics*. The second approach looked for solutions transforming the original problem in another, equivalent one. This developed into *Analytic Mechanics* or *Rational Mechanics*.
In the last 100 years, the study of the *N*-body problem lead to the development of another area of inquiry: *Stellar Dynamics*.

Background

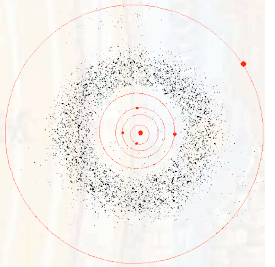
- Pierre Simon de Laplace investigated the stability of the Solar System and published in 1799 his classic work: *Mécanique Celeste*.

- Leonhard Euler seems to have been the first to study the three-body problem as a pure mathematical problem. He found a solution of the *restricted 3-body problem* in which all particles move along a straight path.

- Joseph Louis Lagrange studied the motion of the Moon and gave a detailed analysis of the orbits of Jupiter's satellites. He confirmed Euler's solution for the general 3-body problem and found an additional solution where the 3 bodies move forming an equilateral triangle.

- Karl Friedrich Gauss developed a general method to determine orbits based on only three observations and found the asteroid Ceres, which had been lost when it passed behind the Sun.

- Henri Jules Poincaré studied very thoroughly the 3-body problem. The methods he introduced to study its stability lead to the development of Topology and to the discovery of the concept of Chaos.



Background

The *N*-body problem is of great interest for several sciences

(i) $M = T^*W$ with canonical symplectic structure,

$$W = \mathbf{R}^3 \times \mathbf{R}^3 \setminus \Delta, \qquad \Delta = \{(\mathbf{q}, \mathbf{q}) \mid \mathbf{q} \in \mathbf{R}^3\};$$

(ii) $m \in M$ (initial conditions);

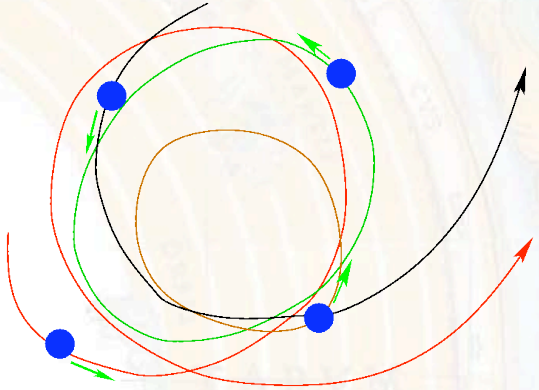
(iii) $\mu \in \mathbf{R}$, $\mu > 0$ (mass ratio);

(iv) $H^\mu \in F(M)$ given by

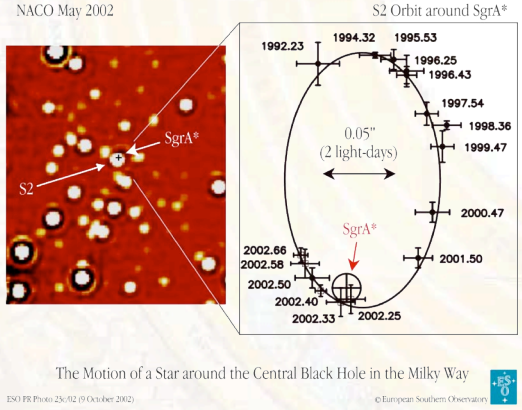
$$H^\mu(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}') = \frac{\| \mathbf{p} \|^2}{2\mu} + \frac{\| \mathbf{p}' \|^2}{2\mu} - \frac{1}{\| \mathbf{q} - \mathbf{q}' \|},$$

where $\mathbf{q}, \mathbf{q}' \in \mathbf{R}^3$, $\mathbf{p}, \mathbf{p}' \in (\mathbf{R}^3)^*$, and $\| \ \|$ denotes the Euclidian norm in $\mathbf{R}^3$.

Mathematics



Physics



Astronomy

```
do i = 1, N
   F_i = 0
end do

do i = 1, N - 1
do j = 1 + 1, N
   F_ij = (G m_i m_j / r_ij^3) r_ij
   F_i = F_i - F_ij
   F_j = F_j - F_ij
end do
end do

do i = 1, N
   v_i(t + Δt) = v_i(t) + (F_i / m_i) Δt
   r_i(t + Δt) = r_i(t) + v_i Δt
end do
```

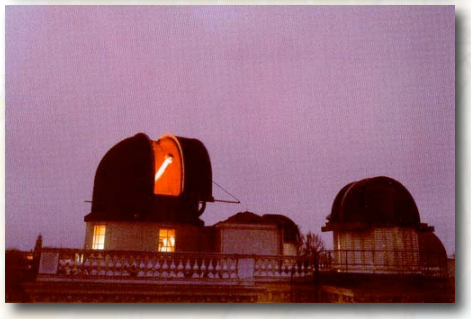
Computer Science

Background

The *N*-body problem and the quest for power among European powers

The great European powers established national observatories and promoted the development of mathematics and astronomy, trying to solve a problem of large strategic importance: navigation.

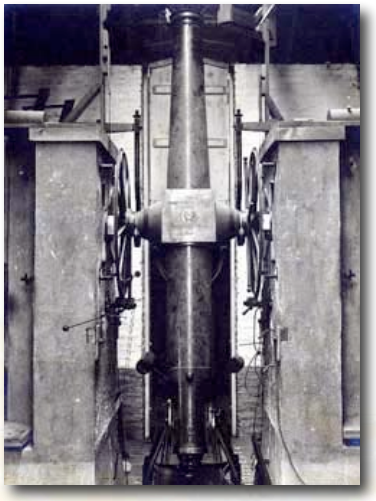
The search for a solution split in two camps: Some turned to mechanical devices to measure time accurately and thus determine geographical position, while others kept looking at the sky for the perfect clock.

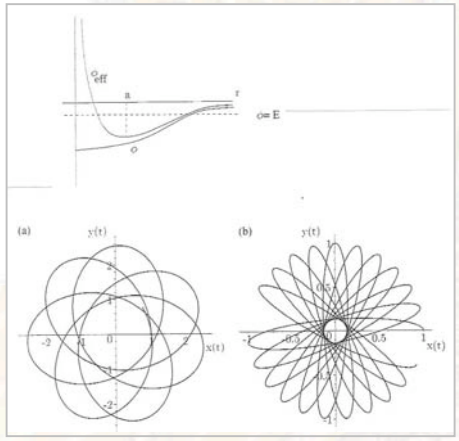


Background

The *N*-body problem and the quest for power among European powers

The second approach motivated the study of the *N*-body problem and brought an enormous development of Mathematics and the Physical Sciences.





Background

So, it is fair to say that the N-body problem is one of the most fruitful problems in the history of science.

The nature of gravity

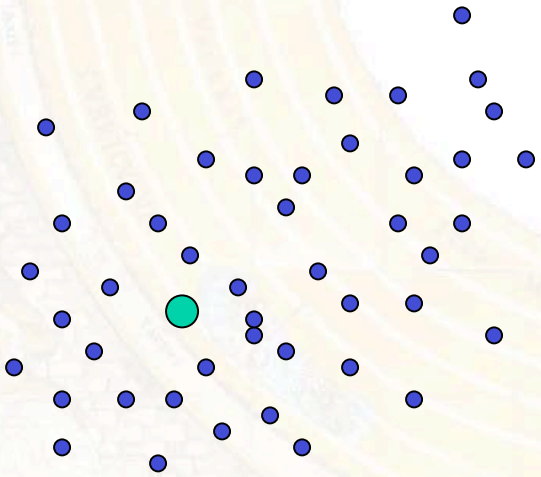
Although gravity does not have an intrinsic length scale, its application to discrete systems does introduce a characteristic scale.

The nature of gravity

Although gravity does not have an intrinsic length scale, its application to discrete systems does introduce a characteristic scale.

Let's imagine a test particle at the center of a spherical system that contains a huge number of point particles.

It is clear that the net average force on the test particle will be zero, since the distribution of matter is symmetric around it: there is, on average, the same number of particles pulling on each direction.

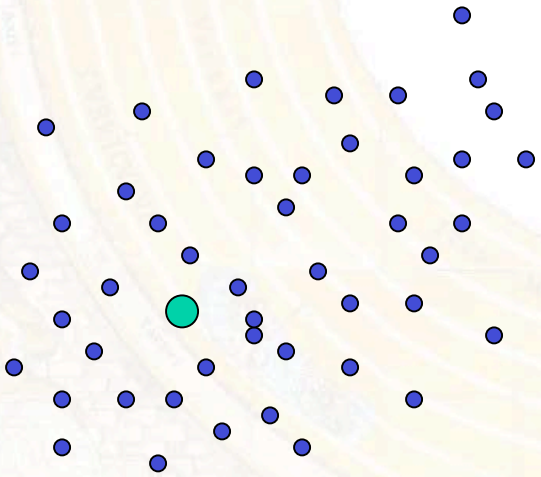


The nature of gravity

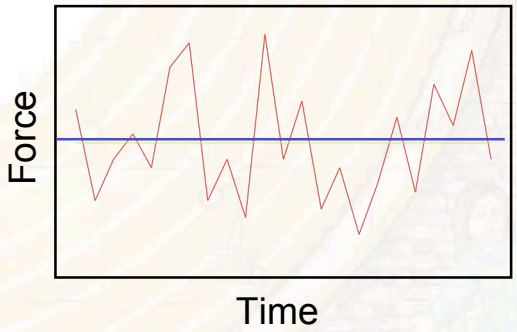
Although gravity does not have an intrinsic length scale, its application to discrete systems does introduce a characteristic scale.

Let's imagine a test particle at the center of a spherical system that contains a huge number of point particles.

It is clear that the net average force on the test particle will be zero, since the distribution of matter is symmetric around it: there is, on average, the same number of particles pulling on each direction.



The instantaneous force, however, will fluctuate around zero, due to the discrete nature of the mas distribution.



The nature of gravity

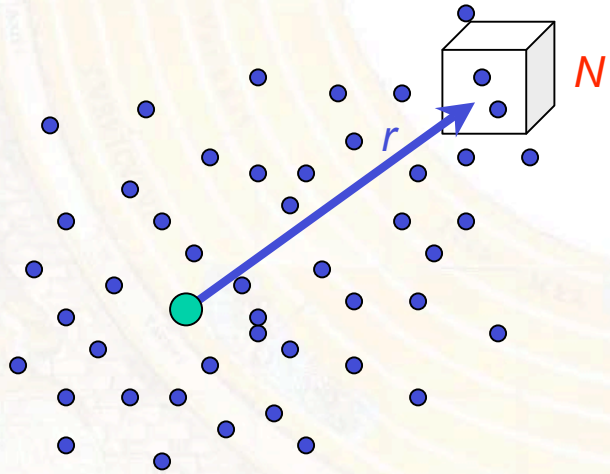
Although gravity does not have an intrinsic length scale, its application to discrete systems does introduce a characteristic scale.

If we center our attention to a particular region which contains *N* stars at a distance *r* from us, it is clear that the force due to it is proportional to:

$$F \propto \frac{N}{r^2}$$

Due to the finite discrete nature of the system, there will be fluctuations in the force of a magnitude given by:

$$\delta F \propto \frac{\delta N}{r^2} = \frac{\sqrt{N}}{r^2}$$



The nature of gravity

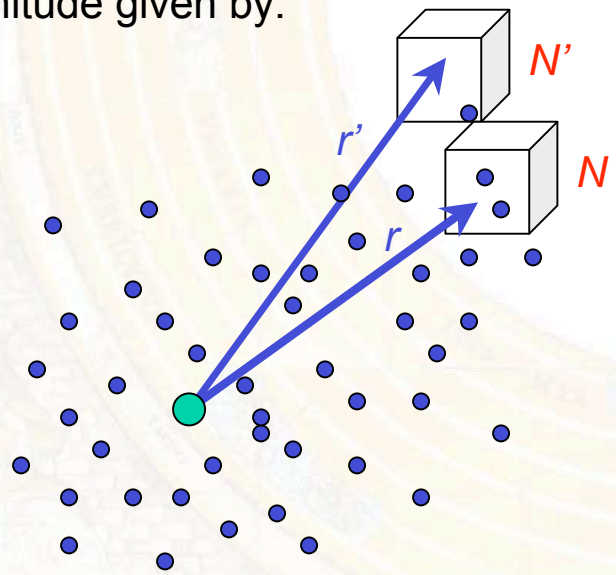
Although gravity does not have an intrinsic length scale, its application to discrete systems does introduce a characteristic scale.

If we center our attention to a particular region which contains *N* stars at a distance *r* from us, it is clear that the force due to it is proportional to:

$$F \propto \frac{N}{r^2}$$

Due to the finite discrete nature of the system, there will be fluctuations in the force of a magnitude given by:

$$\delta F \propto \frac{\delta N}{r^2} = \frac{\sqrt{N}}{r^2}$$



If we now look at another region at a distance *r'* and ask that it exerts the same average force, it is clear that it must contain a number of stars given by:

$$N' \propto r'^2$$

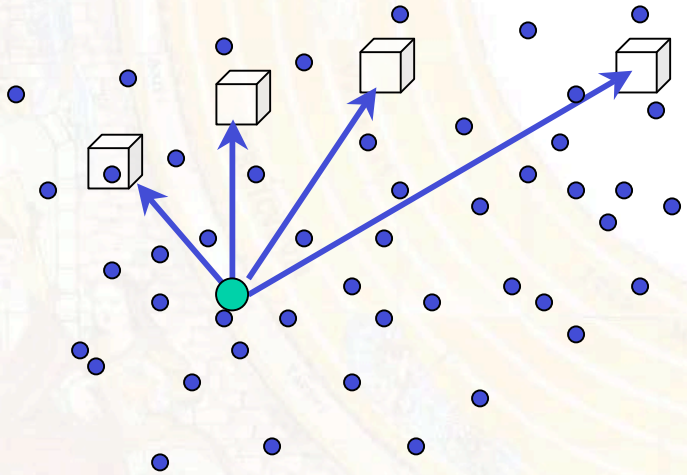
and its fluctuations will be given by,

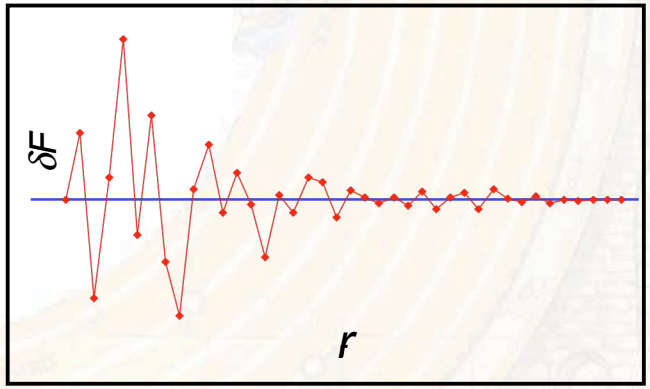
$$\delta F' \propto \frac{\sqrt{N'}}{r^2} \propto \frac{1}{r'}$$

The nature of gravity

$$\Delta F \propto \frac{1}{r}$$

This is a very important result, it means that if we split a discrete system into parcels, such that each one exerts on average the same force on the test particle, the fluctuations in their forces will shrink wit distance as *1/r*.



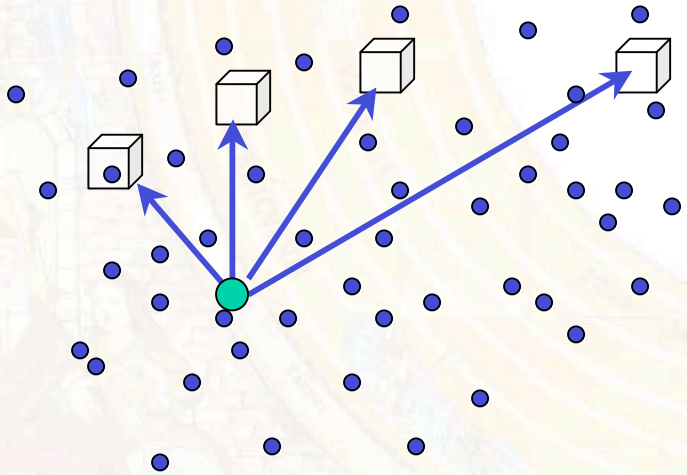


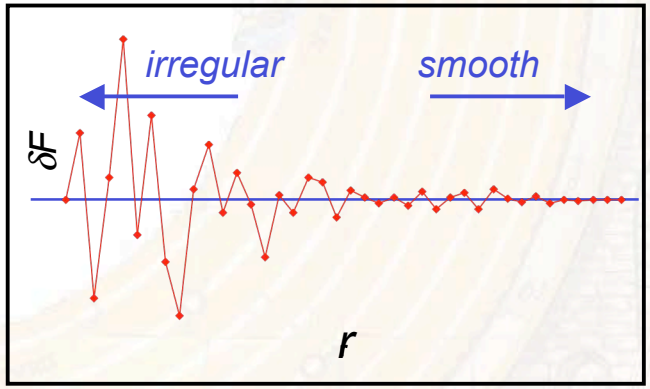
The nature of gravity

$$\Delta F \propto \frac{1}{r}$$

This is a very important result, it means that if we split a discrete system into parcels, such that each one exerts on average the same force on the test particle, the fluctuations in their forces will shrink with distance as *1/r*.

We can then split the gravitational influence into two asymptotic regimes: a fluctuating part due to close neighbors and a smooth part due to distant particles.





The nature of gravity

The characteristic time in which the irregular part dominates is called the *collisional time* and it can be shown that it is of the form:

$$t_{col} \propto \left(\frac{R}{<v^2>}\right) N / \log(N)$$

On the other hand, the natural time unit in a gravitational system is the so called *dynamical time* in which particles cross the system:

$$t_{dyn} \propto \left(\frac{R}{<v^2>}\right)$$

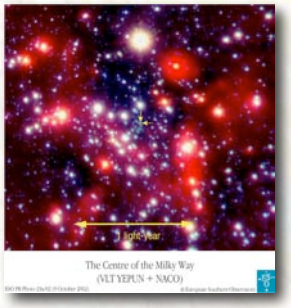
Their ratio measures the importance of the irregular part with respect to the smooth one:

$$t_{col} / t_{dyn} \propto N / \log(N)$$

The nature of gravity

The smaller this ratio, the more the system will be dominated by the irregular regim where the behavior of the force is dominated by collisions with close neighbors. Thu we can split self-gravitating N-body systems in two types: *collisional* and *collisionles systems*; the fundamental difference between them being set by the number of particles.

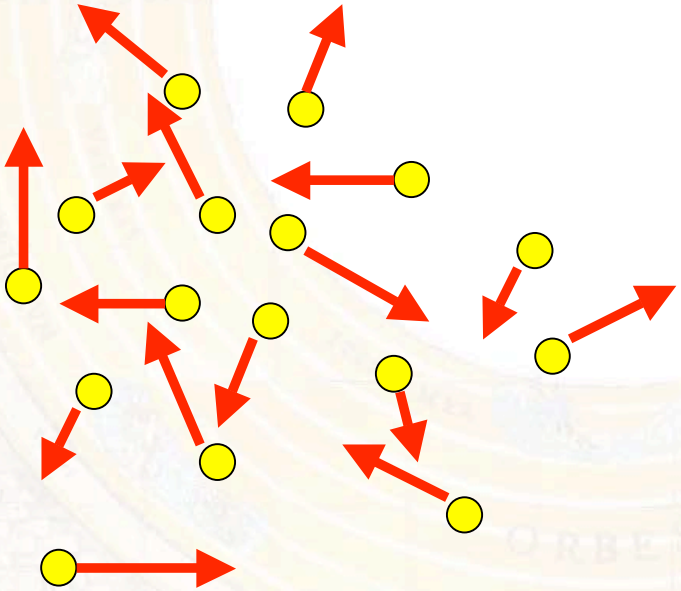
System	N	t_{col} (10^6 years)	t_{dyn} (10^6 years)	t_{col}/t_{dyn}
Stellar group	≤ ~10	≤1	≤1	~1
Globular cluster	~10^6	≤10^3	~10	~10^2
Galaxy	~10^{11}	~10^7	10^2	~10^5
Cluster of galaxies	10^2-10^3	~10^3	10^3	~10



The nature of gravity

The fact that the importance of collisions *decreases* as *N increases* is a bit counterintuitive, so it deserves a bit of thought

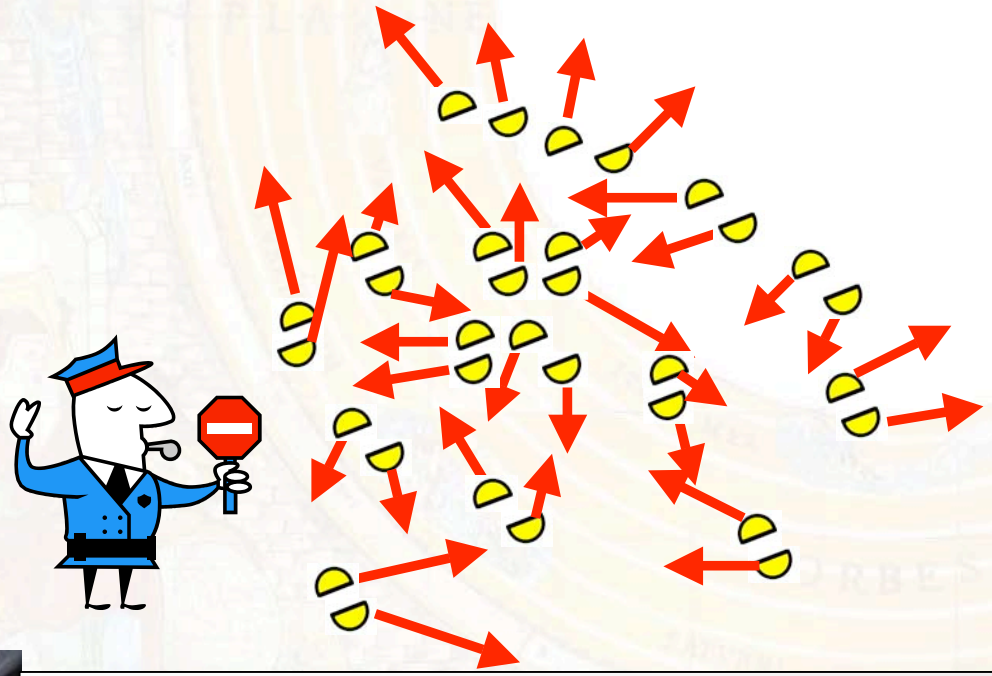
Let us imagine a self-gravitating system of *N* particles in virial equilibrium, it is clear that as time passes some collisions will occur.



The nature of gravity

The fact that the importance of collisions *decreases* as *N increases* is a bit counterintuitive, so it deserves a bit of thought

Let us now imagine that we split each particle in two, thus doubling the number of particles in the system without perturbing the virial equilibrium. It is clear that the number of collisions per unit time will double.



So, how can we say that the importance of collisions diminishes?

The nature of gravity

What matters is not the *number* of collisions, but the *effect* they have.

In the case of cars, collisions imply physical contact, and the faster the relative velocity, the larger the damage produced.

In the case of stars, we have long range interactions, where the larger the relative velocity, the shorter the interaction time, and less damage will occur.

The nature of gravity

In the case of our hypothetical system with *N* particles, the strength of the gravitational force goes as the product of the masses divided by their separation:

$$F \propto (m \bullet m) / r^2$$

When we split each particle in two, we doubled the number of particles (and collisions), bu the mass was halved, so the numerator got reduced by a factor of 1/4:

$$m \rightarrow m/2 \;\Rightarrow\; F \rightarrow F/2^2$$

It's true that the mean separation between particles got reduced as well, which drives the effect in the other direction, but this effect can't overcome the mass effect:

$$r \propto N^{-1/3} \quad \Rightarrow \quad F \rightarrow F \times (1/2^2) \bullet (2^{2/3}) = F \times 2^{-4/3}$$

Even after multiplying by 2 to account for the larger number of collisions, the net effect shrin

The basic parts of an N-body code

The two most important parts of an N-Body code are:

- A force routine that computes forces.
- An integrator that integrates the equations of motion.

The choices made for these two parts largely define the merits and limits of an N-body code.

The basic parts of an N-body code

Force calculators

The most important feature that defines the type of force calculator is the collisionality of the system. Other properties, like overall symmetry, are important too.

Less collisional →

PP	P^3M	Tree	MFC	PM	Fourier

The basic parts of an N-body code

Integrators

Integrators are defined to a large extent by the numerical precision required. The dynamical range of the timescales as well as restrictions on computer memory or computing speed are also relevant.

Fixed time-step *vs* variable time step
Low order *vs* high order schemes
Symplectic *vs* conventional integrators

The limits of an N-body simulation

We must never forget that a given N-body simulation, however large and carefully planned, is always an approximation:

- Limitations of the integrator
- Approximations of the force calculator
- Finite dynamical range
- Issues related to coarseness of the simulation compared to real systems
- Non-equilibrium initial conditions
- Convergence

You would be surprised to see the number of published works that are flawed, not for limitations of computer technology, but by poor design.

Outline of the school

Topics and lecturers:

- Unix & Nemo (Jean Charles Lambert & Peter Teuben)
- Initial conditions (Luis Aguilar)
- Force routines (Walter Dehnen)
- Integrators (Peter Teuben)
- The Astrophysics behind (Lia Athanassoula)
- Cosmology (Julio Navarro)

Jean Charles Lambert

Peter Teuben

Luis Aguilar

Walter Dehnen

Lia Athanassoula

Julio Navarro

Outline of the school

Topics and lecturers:

- Unix & Nemo (Jean Charles Lambert & Peter Teuben)
- Initial conditions (Luis Aguilar)
- Force routines (Walter Dehnen)
- Integrators (Peter Teuben)
- The Astrophysics behind (Lia Athanassoula)
- Cosmology (Julio Navarro)

Structure of the school:

- Regular morning lectures
- Afternoon computer labs
- Evening conferences:
  - A. Bosma, I. Puerari, O. Valenzuela, P. Colín, V. Avila

Hands on experience:

- Basic setup
- Simple exercises
- School projects

My favorite N-body simulation

I can not finish without telling you about my favorite simulation

The year is 1941 and based on surveys like the Shapley-Ames, people were aware of the tendency of luminous galaxies to gather in large conglomerates called clusters of galaxies.

Erik Holmberg conjectured that energy dissipation during galaxy-galaxy encounters was responsible, at least in part, for this tendency.

However, he needed to perform a simulation to gauge the magnitude of the effect. Since at this time no computers were available, at least not to him, he had to do it some other way.

My favorite N-body simulation

Holmberg replaced gravity by light to perform the first published “N-body simulation” in Galactic Dynamics.

THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

VOLUME 94 NOVEMBER 1941 NUMBER 3

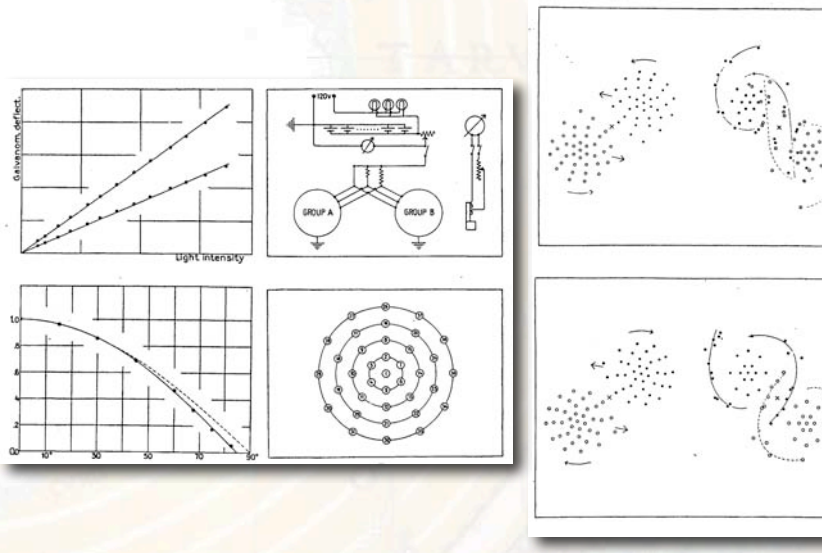
ON THE CLUSTERING TENDENCIES AMONG THE NEBULAE

II. A STUDY OF ENCOUNTERS BETWEEN LABORATORY MODELS OF STELLAR SYSTEMS BY A NEW INTEGRATION PROCEDURE

ERIK HOLMBERG

ABSTRACT

In a previous paper[1] the writer discussed the possibility of explaining the observed clustering effects among extragalactic nebulae as a result of captures. The present investigation deals with the important problem of whether the loss of energy resulting from the tidal disturbances at a close encounter between two nebulae is large enough to effect a capture. The tidal deformations of two models of stellar systems, passing each other at a small distance, are studied by reconstructing, piece by piece, the orbits described by the individual mass elements. The difficulty of integrating the total gravitational force acting upon a certain element at a certain point of time is solved by replacing gravitation by light. The mass elements are represented by light-bulbs, the candle power being proportional to mass, and the total light is measured by a photocell (Fig. 1). The nebulae are assumed to have a flattened shape, and each is represented by 37 light-bulbs. It is found that the tidal deformations cause an increase in the attraction between the two objects, the increase reaching its maximum value when the nebulae are separating, i.e., after the passage. The resulting loss of energy (Fig. 6) is comparatively large and may, in favorable cases, effect a capture. The spiral arms developing during the encounter (Figs. 4) represent an interesting by-product of the investigation. The direction of the arms depends on the direction of rotation of the nebulae with respect to the direction of their space motions.



The End

or rather, the beginning.